gr-qc/9911069

Uniqueness theorems for static black holes in metric–affine gravity

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(February 7, 2008)

Abstract

Using the equivalence theorem for the *triplet ansatz* sector of metric–affine gravity (MAG) theories and the Einstein–Proca system, it is shown that the only static black hole of the triplet sector of MAG is the Schwarzschild solution, under the constraint $(-4\beta_4 + k_1\beta_5/2k_0 + k_2\gamma_4/k_0)/\kappa z_4 \neq 0$ on the coupling constants. For the special case $(-4\beta_4 + k_1\beta_5/2k_0 + k_2\gamma_4/k_0)/\kappa z_4 = 0$, it follows that the only static non–extremal black hole is the Reissner–Nordström one. The results can be extended to exclude also the existence of soliton solutions of the triplet sector of MAG.

PACS numbers: 04.50.+h, 03.50.Kk, 04.70.Bw, 04.70.-s

Typeset using REVTEX

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I. INTRODUCTION

In February 1916, only three months after having achieved the final breakthrough in general relativity, Einstein presented [1], on behalf of Schwarzschild, the first exact solution of his new equations to the Prussian Academy of Sciences. However, it took almost half a century until the geometry of the Schwarzschild space–time was correctly interpreted and its physical significance was fully appreciated. The vacuum Schwarzschild solution describing the end product of gravitational collapse contains a space–time singularity which is hidden within a black hole.

The mathematical theory of black holes has been steadily developing during the last thirty years. One of the most intriguing outcomes is the so–called "*no–hair*" theorem, which states that a black hole in a stationary electrovacuum space–time is uniquely characterized by its mass, angular momentum and electric charge (see Refs. [2–4] for recent reviews on the subject).

Although Einstein gravity is well founded and experimentally very successful on the macroscopic scale, it is of interest to investigate gravity theories which generalize the restricted geometrical structure of Einstein's theory. The theory of the quantum superstring [5], suggests that non–Riemannian features are present on the scale of the Planck length. Since the possibility of testing the predictions of such fundamental theories is very low, due to the high energy levels where new effects may appear, effective theories offer an alternative scenario to perform tests at lower energy levels. It turns out that low–energy dilaton and axi–dilaton interactions are manageable in terms of a non–Riemannian connection that leads to new geometrical structures with particular torsion and nonmetricity fields [6].

More generally, the metric–affine theory of gravity (MAG), which is basically a gauge theory of the four–dimensional affine group, encompasses the dilaton–axion gravity of the effective string models as a subcase [7,8]. MAG has exact solutions with novel features [9–12] (For a review on exact solutions in MAG see Ref. [13] and references therein). These solutions and a possible necessary symmetry reduction process may pave a way to understand how the Riemann–Einstein structure emerges in gravitational gauge theory.

In MAG, the nonmetricity, torsion and curvature are dynamical variables — field strengths — which together with the coframe, the metric and the connection potentials provide an alternative description of gravitational physics.

Black holes with non–Riemannian geometrical structures have not been extensively studied in the literature. Tresguerres [12] and Tucker and Wang [14] found Reissner–Nordström–like metrics, in which the place of the electric charge is taken by the *dilation charge*, (gravito–electric charge) related to the Weyl covector, the trace of the nonmetricity, together with a vector part of the torsion, i.e., these solutions carry a *dilation* charge (Weyl charge) and a *spin* charge, but are devoid of any other post–Riemannian "excitations," in particular, they have no tracefree pieces of the nonmetricity. The corresponding Lagrangian needs only a Hilbert–Einstein piece and a segmental curvature squared. Other black hole solutions exciting more post–Riemannian structures have been found by Vlachinsky et al. [15] and Obukhov et al. [16]. However, the most general Reissner–Nordström solution in MAG, endowed with electric and magnetic charges, as well as gravito–electric and gravito–magnetic charges and cosmological constant term is presented in [17].

In this paper, by using the equivalence theorem between the *triplet ansatz* sector of

MAG and the Einstein–Proca systems [18,19], we proof a "*no–hair*" theorem for static black holes in this sector of MAG theories. The result implies that the only static black hole of the triplet ansatz sector of MAG, in vacuum is the Schwarzschild solution and, in electrovacuum is the Reissner–Nordström black hole.

In Section II we review the main aspects of MAG and its *triplet ansatz* sector together with the equivalence theorem to the Einstein–Proca systems. In Section III the black hole configurations in MAG are reviewed and discussed. In Section VI under the assumption that it represents the gravitational field of a black hole, we analyze the field equations for a static configuration . We proof a "no–hair" theorem for the effective Proca field derived from the equivalence theorem. In Section V we discuss the physical significance of our results.

II. MAG IN BRIEF AND THE EQUIVALENCE THEOREM

The *most general parity conserving quadratic* Lagrangian which is expressed in terms of the $4+3+11$ irreducible pieces (see [20,13]) of $Q_{\alpha\beta}$, T^α, $R_\alpha{}^\beta$, respectively, reads:

$$\begin{aligned}
V_{\rm MAG} = \frac{1}{2\kappa}\,\Big[&-a_0\, R^{\alpha\beta}\wedge\eta_{\alpha\beta} + T^\alpha\wedge{}^*\Big(\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\Big)\\
&+\,2\Big(\sum_{I=2}^{4} c_I\,{}^{(I)}Q_{\alpha\beta}\Big)\wedge\vartheta^\alpha\wedge{}^*T^\beta + Q_{\alpha\beta}\wedge{}^*\Big(\sum_{I=1}^{4} b_I\,{}^{(I)}Q^{\alpha\beta}\Big)\\
&+\,b_5\left({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^\alpha\right)\wedge{}^*\left({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_\beta\right)\Big]\\
&-\,\frac{1}{2\rho}\,R^{\alpha\beta}\wedge{}^*\Big(\sum_{I=1}^{6} w_I\,{}^{(I)}W_{\alpha\beta} + w_7\,\vartheta_\alpha\wedge(e_\gamma\rfloor{}^{(5)}W^\gamma{}_\beta)\\
&+\,\sum_{I=1}^{5} z_I\,{}^{(I)}Z_{\alpha\beta} + z_6\,\vartheta_\gamma\wedge(e_\alpha\rfloor{}^{(2)}Z^\gamma{}_\beta) + \sum_{I=7}^{9} z_I\,\vartheta_\alpha\wedge(e_\gamma\rfloor{}^{(I-4)}Z^\gamma{}_\beta)\Big)\,.
\end{aligned} \tag{2.1}$$

Here $a_0,\ \ldots,\ a_3,\ b_1,\ \ldots,\ b_5,\ c_2,\ c_3,\ c_4,\ w_1,\ \ldots,\ w_7,\ z_1,\ \ldots,\ z_9$ are dimensionless coupling constants, κ is the standard gravitational constant, and ρ is the strong gravity coupling constant. We have introduced in the curvature square term the irreducible pieces of the antisymmetric part $W_{\alpha\beta}\equiv R_{[\alpha\beta]}$ and the symmetric part $Z_{\alpha\beta}\equiv R_{(\alpha\beta)}$ of the curvature two–form. In $Z_{\alpha\beta}$, we have the purely *post*–Riemannian part of the curvature. Note the peculiar cross terms with c_I and b_5. The signature of space–time is $(-,+,+,+)$, the volume four–form $\eta\equiv{}^*1$, and the two–form $\eta_{\alpha\beta}\equiv{}^*(\vartheta_\alpha\wedge\vartheta_\beta)$.

Therefore, space–time is described by a metric–affine geometry with the gravitational *field strengths* nonmetricity $Q_{\alpha\beta}\equiv -Dg_{\alpha\beta}$, torsion $T^\alpha\equiv D\vartheta^\alpha$, and curvature $R_\alpha{}^\beta\equiv d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma\wedge\Gamma_\gamma{}^\beta$. The gravitational field equations

$$DH_\alpha - E_\alpha = \Sigma_\alpha\,, \tag{2.2}$$
$$DH^\alpha{}_\beta - E^\alpha{}_\beta = \Delta^\alpha{}_\beta\,, \tag{2.3}$$

link the *material sources*, the material energy–momentum current Σ_α and the material hypermomentum current $\Delta^\alpha{}_\beta$, to the gauge field *excitations* H_α and $H^\alpha{}_\beta$ in a Yang–Mills–like manner. It is well known [20] that the field equation corresponding to the variable $g_{\alpha\beta}$ is

redundant if (2.2) as well as (2.3) are fulfilled. The excitations can be calculated by partial differentiation,

$$H_\alpha = -\frac{\partial V_{\rm MAG}}{\partial T^\alpha}\,, \quad H^\alpha{}_\beta = -\frac{\partial V_{\rm MAG}}{\partial R_\alpha{}^\beta}\,, \quad M^{\alpha\beta} = -2\frac{\partial V_{\rm MAG}}{\partial Q_{\alpha\beta}}\,, \tag{2.4}$$

whereas the gauge field currents of energy–momentum and hypermomentum, respectively, turn out to be linear in the Lagrangian and in the excitations,

$$E_\alpha \equiv \frac{\partial V_{\rm MAG}}{\partial \vartheta^\alpha} = e_\alpha \rfloor V_{\rm MAG} + (e_\alpha \rfloor T^\beta)\wedge H_\beta + (e_\alpha \rfloor R_\beta{}^\gamma)\wedge H^\beta{}_\gamma + \frac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})M^{\beta\gamma}\,, \tag{2.5}$$

$$E^\alpha{}_\beta \equiv \frac{\partial V_{\rm MAG}}{\partial \Gamma_\alpha{}^\beta} = -\vartheta^\alpha \wedge H_\beta - g_{\beta\gamma}M^{\alpha\gamma}\,. \tag{2.6}$$

Here e_α represents the frame and $\rfloor$ the interior product sign, for details see [20]. In vacuum, the energy–momentum current $\Sigma_\alpha = 0$ and the material hypermomentum current $\Delta^\alpha_\beta = 0$.

A. Triplet ansatz and equivalence theorem

For the torsion and nonmetricity field configurations, we concentrate on the simplest non–trivial case *with* shear. According to its irreducible decomposition [20], the nonmetricity contains two covector pieces, namely ${}^{(4)}Q_{\alpha\beta} = Q\, g_{\alpha\beta}$, with $Q \equiv g^{\alpha\beta}\, Q_{\alpha\beta}/4$, the dilation piece, and

$${}^{(3)}Q_{\alpha\beta} = \frac{4}{9}\left(\vartheta_{(\alpha}e_{\beta)}\rfloor\Lambda - \frac{1}{4}g_{\alpha\beta}\Lambda\right)\,, \quad \text{with} \quad \Lambda \equiv \vartheta^\alpha e^\beta \rfloor \not{Q}_{\alpha\beta}\,, \tag{2.7}$$

a proper shear piece. Accordingly, our ansatz for the nonmetricity reads

$$Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}\,. \tag{2.8}$$

The torsion, in addition to its tensor piece, encompasses a covector and an axial covector piece. Let us choose only the covector piece as non–vanishing:

$$T^\alpha = {}^{(2)}T^\alpha = \frac{1}{3}\,\vartheta^\alpha \wedge T\,, \quad \text{with} \quad T \equiv e_\alpha \rfloor T^\alpha\,. \tag{2.9}$$

Thus we are left with the three non–trivial one–forms Q, Λ, and T.

The Lagrangian (2.1) is very complicated, in particular on account of its curvature square pieces. Therefore we have to restrict its generality in order to stay within manageable limits. Our ansatz for the nonmetricity is expected to require a nonvanishing post–Riemannian term quadratic in the segmental curvature. Accordingly, let be given the gauge Lagrangian (2.1) with $w_1 = \ldots = w_7 = 0$, $z_1 = \ldots = z_3 = z_5 = \ldots = z_9 = 0$, that is, only z_4 is allowed to survive, i.e., the *segmental* curvature squared

$$-\frac{z_4}{8\rho}\, R_\alpha{}^\alpha \wedge {}^*R_\beta{}^\beta = -\frac{z_4}{2\rho}\, dQ \wedge {}^*dQ \tag{2.10}$$

is the only surviving strong gravity piece in $V_{\rm MAG}$.

We assume the following ansatz, the so–called *triplet ansatz* for our triplet of one forms (2.8) and (2.9):

$$Q = k_0\,\phi\,, \qquad \Lambda = k_1\,\phi\,, \qquad T = k_2\,\phi\,, \tag{2.11}$$

where $k_0 \equiv 4\alpha_2\beta_3 - 3\gamma_3^2$, $k_1 \equiv 9(\alpha_2\beta_5/2 - \gamma_3\gamma_4)$, $k_2 \equiv 3(4\beta_3\gamma_4 - 3\beta_5\gamma_3/2)$, and $\alpha_2 = a_2 - 2a_0$, $\beta_3 = b_3 + a_0/8$, $\beta_4 = b_4 - 3a_0/8$, $\beta_5 = b_5 - a_0$, $\gamma_3 = c_3 + a_0$, $\gamma_4 = c_4 + a_0$. In other words, we assume that the triplet of one–forms are proportional to each other [16,13,18,19,21].

The triplet ansatz (2.11) reduces the MAG field equations (2.2)–(2.3) to an effective Einstein–Proca system [18,19]:

$$\frac{a_0}{2}\,\eta_{\alpha\beta\gamma} \wedge \tilde{R}^{\beta\gamma} = \kappa\,\Sigma_\alpha^{(\phi)}, \tag{2.12}$$

$$d\,{}^*H + m^2\,{}^*\phi = 0, \tag{2.13}$$

with respect to the metric g, and the Proca 1–form ϕ. Here the tilde$\tilde{}$ denotes the Riemannian part of the curvature,

$$\Sigma_\alpha^{(\phi)} \equiv \frac{z_4 k_0^2}{2\rho}\left\{[(e_\alpha\rfloor H) \wedge {}^*H \;-\; (e_\alpha\rfloor\,{}^*H) \wedge H] \right. \\ \left. +\, m^2\;[(e_\alpha\rfloor\phi) \wedge {}^*\phi \;+\; (e_\alpha\rfloor\,{}^*\phi) \wedge \phi]\right\}\,, \tag{2.14}$$

is the energy–momentum current of the Proca field ϕ, $H \equiv d\phi$, and

$$m^2 \equiv \frac{1}{\kappa z_4}\left(-4\beta_4 + \frac{k_1}{2k_0}\beta_5 + \frac{k_2}{k_0}\gamma_4\right), \tag{2.15}$$

Therefore, as mentioned above, given the triplet ansatz MAG becomes an effective Einstein–Proca system. Moreover, by setting $m = 0$ the system acquires the constraint $\beta_4 = (k_1\beta_5/2 + k_2\gamma_4)/4k_0$ among the coupling constants of the Lagrangian (2.1), and it reduces to the Einstein–Maxwell system, cf. Ref. [13].

Thus, the triplet ansatz sector of a MAG theory becomes equivalent to the Einstein–Proca system of differential equations. This was first shown for a certain 3–parameter Lagrangian by Dereli et al. [22] and extended to a 6–parameter Lagrangian by Tucker and Wang [19]. The situation was eventually clarified for a fairly general 11–parameter Lagrangian by Obukhov et al. [18]. In the next sections we will make use of this equivalence theorem in order to proof the “no–hair” theorem for static black–hole configurations in MAG.

III. BLACK HOLE CONFIGURATIONS IN MAG

The search for black hole solutions in MAG began with Tresguerres [12,23], who found the first static spherically symmetric solutions with a non–vanishing *shear charge*, i.e., the solutions are additionally endowed with a traceless part of the nonmetricity. The metric of Tresguerres solution is the *Reissner–Nordström metric* of general relativity with cosmological constant but the place of the electric charge is taken by the *dilation charge* (gravito–electric charge) which is related to the trace of the nonmetricity, the Weyl covector.

The Tresguerres solutions carry, besides the above–mentioned dilation and shear charges (related to the trace and traceless pieces of the nonmetricity, respectively), a *spin charge* related to the torsion of space–time. Therefore, beyond the Reissner–Nordström metric, the following post–Riemannian degrees of freedom are excited in the Tresguerres solutions: *two* pieces of the nonmetricity, namely the Weyl covector ${}^{(4)}Q_{\alpha\beta}$ and the traceless piece ${}^{(2)}Q_{\alpha\beta}$, and all *three* pieces of the torsion ${}^{(1)}T^{\alpha}$, ${}^{(2)}T^{\alpha}$, ${}^{(3)}T^{\alpha}$. The first solution [12], requires in the Lagrangian weak gravity terms and, for strong gravity, the curvature square pieces with $z_4 \neq 0$, $w_3 \neq 0$, $w_5 \neq 0$, i.e., with Weyl's segmental curvature, the curvature pseudoscalar, and the antisymmetric Ricci.

Beside the two dilation–shear solutions, Tresguerres [12] and Tucker and Wang [19] found Reissner–Nordström–like metrics together with a non–vanishing Weyl covector, ${}^{(4)}Q^{\alpha\beta} \neq 0$, and a vector part of the torsion, ${}^{(2)}T^{\alpha} \neq 0$, i.e., these solutions carry a *dilation* charge (a Weyl charge) and a *spin* charge, but are devoid of any other post–Riemannian "excitations", in particular, they have no tracefree pieces $\not{Q}_{\alpha\beta}$ of the nonmetricity. The corresponding Lagrangian needs only a Hilbert–Einstein piece ($a_0 = 1$) and a segmental curvature squared with $z_4 \neq 0$. The same has been proved for the Tresguerres dilation solution [12] (see footnote 4 of [16]).

In the framework of the triplet ansatz (2.11), a Reissner–Nordström–like metric with a strong gravito–electric charge could successfully be used [15] and a constraint on the coupling constants had to be imposed. Thus the structure of this *triplet* solution is reminiscent of the Tresguerres dilation–shear solutions. Moreover, only the piece with $z_4 \neq 0$ of the *curvature square pieces* in the gauge Lagrangian V_{MAG} is required. All others do not contribute. This result was generalized to an *axially symmetric* solution [15] based on the *Kerr–Newman* metric, with the same kind of charges. A generalized Reissner–Nordström solution in MAG endowed with electric, magnetic, strong gravito–electric and strong gravito–magnetic charges and cosmological constant is presented in [17].

IV. UNIQUENESS THEOREM FOR STATIC BLACK HOLES IN METRIC–AFFINE GRAVITY

For $m \neq 0$ the equivalence theorem [18,19] establishes that the triplet ansatz sector of MAG is described by the Einstein–Proca system (2.12)–(2.13). In this section we will proof that the effective Proca field ϕ_μ vanishes in the domain of outer communications $\langle\langle \mathcal{J} \rangle\rangle$ of a static black hole. We would like to point out that the original proof on the non–existence of massive–Proca fields in the presence of static black holes is due to Bekenstein [24]. In this paper we improve the original demonstration, basically in the arguments concerning the event horizon, which are the more involved.

Using the standard component notation the Einstein–Proca equations (2.12)–(2.13) can be written as

$$\frac{4\pi}{\widetilde{\kappa}} R_{\mu\nu} = H_{\mu}{}^{\alpha} H_{\nu\alpha} + m^2 \phi_\mu \phi_\nu - \frac{1}{4} g_{\mu\nu} H_{\alpha\beta} H^{\alpha\beta}\,, \tag{4.1}$$

$$\nabla_\beta H^{\beta\alpha} = m^2 \phi^\alpha, \tag{4.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, $H_{\mu\nu} \equiv 2\nabla_{[\mu}\phi_{\nu]}$ is the field strength of the Proca field ϕ_μ, and $\widetilde{\kappa} \equiv \kappa z_4 k_0^2/\rho a_0$.

In a static black hole, the Killing field $\boldsymbol{k}$ coincides with the null generator of the event horizon $\mathcal{H}^+$ and is time–like and hypersurface orthogonal in all the domain of outer communications $\langle\langle \mathscr{J} \rangle\rangle$. This allow us to choose, by simply connectedness of $\langle\langle \mathscr{J} \rangle\rangle$ [25], a global coordinates system (t, x^i), $i = 1, 2, 3$, in all $\langle\langle \mathscr{J} \rangle\rangle$ [26], such that $\boldsymbol{k} = \boldsymbol{\partial/\partial t}$ and the metric reads

$$\boldsymbol{g} = -V\boldsymbol{dt}^2 + \gamma_{ij}\boldsymbol{dx}^i\boldsymbol{dx}^j, \tag{4.3}$$

where V and $\boldsymbol{\gamma}$ are t–independent, $\boldsymbol{\gamma}$ is positive definite in all $\langle\langle \mathscr{J} \rangle\rangle$, and V is positive in all $\langle\langle \mathscr{J} \rangle\rangle$, and vanishes in $\mathcal{H}^+$. From (4.3) it can be noted that staticity is equivalent to the existence of a time–reversal isometry $t \mapsto -t$.

We will assume that the Proca field shares the same symmetries as the metric one, namely, it is stationary, $\pounds_{\boldsymbol{k}}\boldsymbol{\phi} = 0$. The staticity of the metric is extended to the Proca field ϕ^α and the Proca equations (4.2), i.e., they are invariant under time–reversal transformations. The time–reversal invariance of Proca equations (4.2) requires that, in the coordinates (4.3), ϕ^t and H^{ti} remain unchanged while ϕ^i and H^{ij} change their sign, or the opposite scheme, i.e., ϕ^t and H^{ti} change sign as long as ϕ^i and H^{ij} remain unchanged under time reversal [24]. Therefore ϕ^i and H^{ij} must vanish in the first case, and ϕ^t and H^{ti} vanish in the second one. Hence, time–reversal invariance implies the existence of two separate cases: a purely gravito–electric case (I) and a purely gravito–magnetic case (II).

Now we are ready to proof the "*no–hair*" theorem for the effective Proca field. Let $\mathcal{V} \subset \langle\langle \mathscr{J} \rangle\rangle$ be the open region bounded by the space–like hypersurface Σ, the space–like hypersurface Σ' and the pertinent portions of the horizon $\mathcal{H}^+$, and the spatial infinity i^o. The space–like hypersurface Σ' is obtained by shifting each point of Σ a unit parametric value along the integral curves of the Killing field $\boldsymbol{k}$. Multiplying the Proca equations (4.2) by ϕ_μ and integrating by parts over $\mathcal{V}$ using the Gauss theorem, one obtains

$$\left[\int_{\Sigma'} - \int_{\Sigma} + \int_{\mathcal{H}^+\cap\overline{\mathcal{V}}} + \int_{i^o\cap\overline{\mathcal{V}}}\right] \phi_\alpha H^{\beta\alpha} d\Sigma_\beta = \int_{\mathcal{V}} \left(\frac{1}{2}H_{\alpha\beta}H^{\alpha\beta} + m^2\phi_\alpha\phi^\alpha\right) dv. \tag{4.4}$$

The boundary integral over Σ' cancels out the corresponding one over Σ, since Σ' and Σ are isometric hypersurfaces. The boundary integral over $i^o \cap \overline{\mathcal{V}}$ vanishes by the usual Yukawa fall–off of the massive fields at infinity. We will show that the integrand of the remaining boundary integral at the portion of the horizon $\mathcal{H}^+ \cap \overline{\mathcal{V}}$ also vanishes. To achieve this goal we use the standard measure at the horizon $d\,\Sigma_\beta = 2n_{[\beta}l_{\mu]}l^\mu d\sigma$ [27], where $\boldsymbol{l}$ is the null generator of the horizon, $\boldsymbol{n}$ is the other future–directed null vector ($n_\mu l^\mu = -1$), orthogonal to the space–like cross sections of the horizon, and $d\sigma$ is the surface element. The standard measure follows from choosing the natural volumen 3–form at the horizon, i.e., $\boldsymbol{\eta_3} = {}^*(\boldsymbol{n} \wedge \boldsymbol{l}) \wedge \boldsymbol{l}$. By using the quoted measure the integrand over the horizon can be written as

$$\phi_\alpha H^{\beta\alpha} d\Sigma_\beta = \left(\phi_\alpha H^{\beta\alpha} l_\beta + \phi_\alpha H^{\beta\alpha} n_\beta\, l_\mu l^\mu\right) d\sigma\,. \tag{4.5}$$

In order to show the vanishing of the last integrand it is sufficient to prove that the following quantities on the right hand side of (4.5) are such that: $\phi_\alpha H^{\beta\alpha} l_\beta$ vanishes and $\phi_\alpha H^{\beta\alpha} n_\beta$ remains bounded at the horizon. The behavior of this quantities at the horizon can be

established by studying some invariants constructed from the curvature. By using Einstein equations (4.1), we obtain,

$$\frac{4\pi}{\widetilde{\kappa}} R = m^2 \phi_\mu \phi^\mu, \tag{4.6}$$

$$\frac{16\pi^2}{\widetilde{\kappa}^2} R_{\mu\nu} R^{\mu\nu} = 3H^2 + 4I^2 + \left(H - m^2 \phi_\mu \phi^\mu\right)^2 + 2m^2 H_\mu{}^\alpha \phi^\mu H_{\nu\alpha} \phi^\nu, \tag{4.7}$$

where $H \equiv H_{\alpha\beta} H^{\alpha\beta}/4$, and $I \equiv {}^*H_{\alpha\beta} H^{\alpha\beta}/4$. Since the horizon is a smooth surface the left hand sides of the above equations are bounded on it. From (4.6) it follows that $\phi_\mu \phi^\mu$ is bounded at the horizon. The last term in (4.7) is non–negative in both cases (I) and (II), the remaining terms are also non–negative, and consequently each one is bounded at the horizon, in particular the invariants H and I. Other invariants can be built from the Ricci curvature (4.1) by means of $\boldsymbol{l}$ and $\boldsymbol{n}$, which are well–defined smooth vector fields on the horizon. The first invariant reads

$$\frac{4\pi}{\widetilde{\kappa}} R_{\mu\nu} n^\mu n^\nu = J_\mu J^\mu + m^2 (\phi_\mu n^\mu)^2 - n_\mu n^\mu H\,, \tag{4.8}$$

where $J^\mu \equiv H^{\mu\nu} n_\nu$. The last term above vanishes because the bounded behavior of the invariant H. Since $\boldsymbol{J}$ is orthogonal to the null vector $\boldsymbol{n}$, it must be space–like or null ($J_\mu J^\mu \geq 0$), therefore each one of the remaining terms on the right hand side of (4.8) must be bounded. The next invariant to be considered, which vanishes at the horizon by applying the Raychaudhuri equation to the null generator [28] reads

$$0 = \frac{4\pi}{\widetilde{\kappa}} R_{\mu\nu} l^\mu l^\nu = D_\mu D^\mu + m^2 (\phi_\mu l^\mu)^2 - l_\mu l^\mu H\,, \tag{4.9}$$

where $D^\mu \equiv H^{\mu\nu} l_\nu$ is the gravito–electric field at the horizon. Once again the bounded behavior of the invariant H can be used to achive the vanishing of the last term of (4.9). Since $\boldsymbol{D}$ is orthogonal to the null generator $\boldsymbol{l}$, it must be space–like or null ($D_\mu D^\mu \geq 0$), consequently each term on the right hand side of (4.9) vanishes independently, which implies that $\phi_\mu l^\mu = 0$ and that $\boldsymbol{D}$ is proportional to the null generator $\boldsymbol{l}$ at the horizon, i.e., $\boldsymbol{D} = -(D_\alpha n^\alpha)\,\boldsymbol{l}$. The last studied invariant gives the following relation,

$$\frac{4\pi}{\widetilde{\kappa}} R_{\mu\nu} l^\mu n^\nu - H = (D_\mu n^\mu)^2 + m^2 (\phi_\mu n^\mu)(\phi_\nu l^\nu), \tag{4.10}$$

where it has been used that $\boldsymbol{D} = -(D_\alpha n^\alpha)\,\boldsymbol{l}$. Since $\phi_\mu l^\mu = 0$ and $\phi_\mu n^\mu$ is bounded at the horizon, it follows that the second term on the right hand side of (4.10) vanishes, thus $D_\mu n^\mu$ is bounded at the horizon as consequence of the bounded behavior of the left hand side of (4.10).

Summarizing, the study of the quoted invariants at the horizon leads to the following conclusions: $D_\mu n^\mu$, $\phi_\mu n^\mu$, $\phi_\mu \phi^\mu$, and $J_\mu J^\mu$ are bounded at the horizon, $\phi_\mu\, l^\mu = 0$ and $\boldsymbol{D} = -(D_\alpha n^\alpha)\,\boldsymbol{l}$ in the same region.

Now we are in position to show the fulfillment of the sufficient conditions for the vanishing of the integrand (4.5) over the horizon, i.e., $\phi_\alpha H^{\beta\alpha} l_\beta$ vanishes and $\phi_\alpha H^{\beta\alpha} n_\beta$ remains bounded

at the horizon. Using the definition $D^\mu \equiv H^{\mu\nu} l_\nu$ and that $\boldsymbol{D} = -(D_\alpha n^\alpha)\,\boldsymbol{l}$, we obtain for the first quantity at the horizon

$$\phi_\alpha H^{\beta\alpha} l_\beta = (D_\mu n^\mu)(\phi_\nu l^\nu) = 0, \tag{4.11}$$

where the vanishing follows from the fact that $D_\mu n^\mu$ is bounded and $\phi_\nu\, l^\nu$ vanishes at the horizon.

For the second quantity we note that $\boldsymbol{\phi}$ and $\boldsymbol{J}$ are orthogonal to the null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, respectively. Therefore, $\boldsymbol{\phi}$ must be space–like or proportional to $\boldsymbol{l}$, and $\boldsymbol{J}$ must be space–like or proportional to $\boldsymbol{n}$. Using a null tetrad basis, constructed with $\boldsymbol{l}$, $\boldsymbol{n}$, and a pair of linearly independent space–like vectors, these last ones being tangent to the space–like cross sections of the horizon, the $\boldsymbol{\phi}$ and $\boldsymbol{J}$ vectors can be written as

$$\boldsymbol{\phi} = -(\phi_\alpha n^\alpha)\boldsymbol{l} + \boldsymbol{\phi}^\perp, \tag{4.12}$$

$$\boldsymbol{J} = -(J_\alpha l^\alpha)\boldsymbol{n} + \boldsymbol{J}^\perp, \tag{4.13}$$

where $\boldsymbol{\phi}^\perp$ and $\boldsymbol{J}^\perp$ are the projections, orthogonal to $\boldsymbol{l}$ and $\boldsymbol{n}$, on the space–like cross sections of the horizon. Using (4.12) and (4.13) it is clear that $\phi_\mu \phi^\mu = \phi^\perp_\mu \phi^{\perp\mu}$, and $J_\mu J^\mu = J^\perp_\mu J^{\perp\mu}$, i.e., the contribution to these bounded magnitudes comes only from the space–like sector orthogonal to $\boldsymbol{l}$ and $\boldsymbol{n}$. With the help of (4.12) and (4.13) the other quantity appearing in the integrand (4.5) can be written as

$$\phi_\alpha H^{\beta\alpha} n_\beta = -\phi_\alpha J^\alpha = -(\phi_\alpha n^\alpha)(D_\beta n^\beta) - \phi^\perp_\alpha J^{\perp\alpha}, \tag{4.14}$$

where the identity $J_\alpha l^\alpha = -D_\alpha n^\alpha$ has been used. The first term in (4.14) is bounded because $\phi_\alpha n^\alpha$ and $D_\beta n^\beta$ are bounded. To the second term the Schwarz inequality applies, since $\boldsymbol{\phi}^\perp$ and $\boldsymbol{J}^\perp$ belong to a space–like subspace. Thus, $(\phi^\perp_\alpha J^{\perp\alpha})^2 \le (\phi^\perp_\mu \phi^{\perp\mu})(J^\perp_\nu J^{\perp\nu}) = (\phi_\mu \phi^\mu)(J_\nu J^\nu)$ and since $\phi_\mu \phi^\mu$ and $J_\nu J^\nu$ are bounded at the horizon, the second term of (4.14) is also bounded.

Finally, the vanishing of (4.11) and the bounded behavior of (4.14), and the null character of $\boldsymbol{l}$ at the horizon lead to the vanishing of the integrand (4.5) over the event horizon.

With no contribution from boundary integrals in (4.4) we shall write the volume integral, using the coordinates from (4.3), for each one of the different cases discussed at the beginning of this section.

For the purely gravito–electric case (I) we have

$$\int_{\mathcal{V}} -V\left(\frac{1}{2}\gamma_{ij} H^{ti} H^{tj} + m^2 (\phi^t)^2\right) dv = 0. \tag{4.15}$$

The non–positiveness of the above integrand, which is the sum of squared terms, implies that the integral is vanishing only if H^{ti} and ϕ^t vanish everywhere in $\mathcal{V}$, and hence in all $\langle\!\langle \mathscr{J} \rangle\!\rangle$.

For the purely gravito–magnetic case (II) the volume integral reads as

$$\int_{\mathcal{V}} \left(\frac{1}{2}\gamma_{ik}\gamma_{jl} H^{kl} H^{ij} + m^2 \phi_i \phi^i\right) dv = 0, \tag{4.16}$$

in this case the non–negativeness of the above integrand is responsible for the vanishing of H^{ij} and ϕ^i in all $\langle\!\langle \mathscr{J} \rangle\!\rangle$.

V. DISCUSSION

By using the equivalence theorem for the triplet sector of MAG, one faces in vacuum an effective Einstein–Proca system, where the Proca field is related to the vector pieces of the irreducible decomposition of the nonmetricity and torsion [18,19]. In this framework, we investigate whether or not this sector of MAG leads to new physics, i.e., we analyze the "*no–hair*" problem in the triplet ansatz sector of MAG.

We prove that, for $(-4\beta_4 + k_1\beta_5/2k_0 + k_2\gamma_4/k_0)/\kappa z_4 \neq 0$, the effective field ϕ_μ is trivial in the presence of a static black hole. This result implies that the equations (2.12)–(2.13) reduce to the Einstein–vacuum ones, for which the only static black hole is the Schwarzschild solution [29]. For the special case where $(-4\beta_4 + k_1\beta_5/2k_0 + k_2\gamma_4/k_0)/\kappa z_4 = 0$, the system (2.12)–(2.13) reduces to the Einstein–Maxwell system, and it is well–known that the only static black hole, with non–degenerate horizon, is the Reissner–Nordström one [30] (see Refs. [2,3] for improvements to the original proofs). It is worthwhile to stress the fact that while previous works by Chruściel and Nadirashvili [31], and by Heusler [32], on the establishment of the uniqueness results in the extremal case were not yet conclusive, the problem seems to be settled recently by Chruściel [33], who establishes the uniqueness of the Majundar–Papapetrou black hole in the extremal case.

It is straightforward to generalize our results to static electrovacuum space–times of the triplet sector of MAG, which in view of the equivalence theorem becomes equivalent to an effective Einstein–Proca–Maxwell system, i.e., the only existing static black hole is the true Reissner–Nordström one, endowed with electric and/or magnetic charge as well as gravito–electric and/or gravito–magnetic charges [17]. The proof involves only and additional Maxwell field.

Moreover, the existence of static soliton (particle–like) solutions can be also excluded using the same arguments, since the only change in the proof is that in this case the boundary of the volume $\mathcal{V}$ is only formed by the isometric surfaces Σ and Σ', and a portion of the spatial infinity i^o, i.e., there are no interior boundary corresponding to the event horizon.

ACKNOWLEDGMENTS

We thank Friedrich W. Hehl, for useful discussions and literature hints. This research was partially supported by CONACyT Grants: "Black Holes, Inflation and Matter", 28339E, 3567E, DGAPA–UNAM Grant 121298, and by a fellowship from the Sistema Nacional de Investigadores (SNI).

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